ON AN ELEMENTARY PROBLEM OF CLOSURE ON AN EQUILATERAL HYPERBOLA.

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In a recent number of the *Nouvelles Annales de Mathématiques*[1] Professor P. Appell proposes the following problem concerning a system of quadruples of points on an equilateral hyperbola:

Let A_1, B_1, C_1, D_1 be four points on an equilateral hyperbola; the altitudes of the triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ intersect respectively in their ortho-centers A_2, B_2, C_2, D_2, which also lie on the curve. Thus, from the four points A_1, B_1, C_1, D_1, four new points A_2, B_2, C_2, D_2, are derived; in the same manner from these four other points A_3, B_3, C_3, D_3 are obtained, . . . and so forth. Under what conditions does the quadruple $A_nB_nC_nD_n$ coincide entirely, or in part, with $A_1B_1C_1D_1$?

According to Appell's suggestion one may choose $xy = 1$ as the equation of the equilateral hyperbola and represent it parametrically in the form

$$x = e^t, \qquad y = e^{-t}, \tag{1}$$

with t as the parameter. If b_1, c_1, d_1, a_2 are the abscissas of A_1, C_1, D_1 and the orthocenter of $A_1C_1D_1$, respectively, then, as is well known, $a_2b_1c_1d_1 = -1$. Hence, when α_1, β_1, γ_1, δ_1; α_2, β_2, $\cdots$ are the parameters of A_1, B_1, C_1, D_1; A_2, B_2, $\cdots$, we have, when the congruence sign $\equiv$ refers throughout to modulo $2i\pi$ (where $i = \sqrt{-1}$),

$$\begin{aligned} \alpha_{p+1} + \beta_p + \gamma_p + \delta_p &\equiv i\pi, \\ \beta_{p+1} + \gamma_p + \delta_p + \alpha_p &\equiv i\pi, \\ \gamma_{p+1} + \delta_p + \alpha_p + \beta_p &\equiv i\pi, \\ \delta_{p+1} + \alpha_p + \beta_p + \gamma_p &\equiv i\pi. \end{aligned} \tag{2}$$

Putting $S_n = \alpha_n + \beta_n + \gamma_n + \delta_n$, from (2) is found

$$S_{p+1} + 3S_p \equiv 0, \tag{3}$$

and

$$\begin{aligned} \alpha_{p+1} - \alpha_p + S_p &\equiv i\pi, \\ \beta_{p+1} - \beta_p + S_p &\equiv i\pi, \\ \gamma_{p+1} - \gamma_p + S_p &\equiv i\pi, \\ \delta_{p+1} - \delta_p + S_p &\equiv i\pi. \end{aligned} \tag{4}$$

This is as far as Appell's suggestion goes. From (3) and (4) by recurrence

[1] Vol. XVIII, pp. 41–42 (February, 1918).

α_{n+1} may be expressed in terms of α_1 and S_1 as

$$\alpha_{n+1} = \alpha_1 - \frac{1-(-3)^n}{4} S_1 + n'i\pi, \tag{5}$$

where n' is even or odd with n.

Similar formulas are obtained for the parameters β_{n+1}, γ_{n+1}, δ_{n+1} of the points B_{n+1}, C_{n+1}, D_{n+1}. The points A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} coincide in the same order with A_1, B_1, C_1, D_1, when

$$-\frac{1-(-3)^n}{4} \cdot S_1 + n'i\pi \equiv 0, \tag{6}$$

which, in case of real points, is possible only when either

$$\text{I} \qquad S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv i\pi,$$

or

$$\text{II} \qquad S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv 0.$$

Clearly, in the first case, the quadruple $A_2B_2C_2D_2$ coincides in the same order with $A_1B_1C_1D_1$, and is therefore closed in itself. For case II, from (4) we find $\alpha_2 = \alpha_1 - S_1 + (2\lambda + 1)i\pi$, or $\alpha_2 = \alpha_1 + (2\lambda + 1)i\pi$; similarly

$$\beta_2 = \beta_1 + (2\lambda + 1)i\pi, \qquad \gamma_2 = \gamma_1 + (2\lambda + 1)i\pi, \qquad \delta_2 = \delta_1 + (2\lambda + 1)i\pi.$$

From this follows that the abscissas of A_2, B_2, C_2, D_2 are in the same order opposite in sign to those of A_1, B_1, C_1, D_1, *i. e.*, that the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent and in central symmetry with respect to the center of the hyperbola.

Since the condition for four concyclic points A_1, B_1, C_1, D_1,

$$\begin{vmatrix} e^{2\alpha_1} + e^{-2\alpha_1} & e^{\alpha_1} & e^{-\alpha_1} & 1 \\ e^{2\beta_1} + e^{-2\beta_1} & e^{\beta_1} & e^{-\beta_1} & 1 \\ e^{2\gamma_1} + e^{-2\gamma_1} & e^{\gamma_1} & e^{-\gamma_1} & 1 \\ e^{2\delta_1} + e^{-2\delta_1} & e^{\delta_1} & e^{-\delta_1} & 1 \end{vmatrix}$$

$$= \{e^{-2(\alpha_1+\beta_1+\gamma_1+\delta_1)} - e^{-(\alpha_1+\beta_1+\gamma_1+\delta_1)}\} \cdot \begin{vmatrix} e^{3\alpha_1} & e^{2\alpha_1} & e^{\alpha_1} & 1 \\ e^{3\beta_1} & e^{2\beta_1} & e^{\beta_1} & 1 \\ e^{3\gamma_1} & e^{2\gamma_1} & e^{\gamma_1} & 1 \\ e^{3\delta_1} & e^{2\delta_1} & e^{\delta_1} & 1 \end{vmatrix} = 0$$

is evidently satisfied in case II, the original quadrangle $A_1B_1C_1D_1$, and consequently also the quadrangle $A_2B_2C_2D_2$ is concyclic. This follows also from the fact that the product of the abscissas of four concyclic points on the hyperbola (1) is $+1$. From (4) follows further that

$$\alpha_3 = \alpha_1 + 2k\pi, \qquad \beta_3 = \beta_1 + 2k\pi, \qquad \gamma_3 = \gamma_1 + 2k\pi, \qquad \delta_3 = \delta_1 + 2k\pi,$$

so that $A_3B_3C_3D_3$ coincides in the same order with $A_1B_1C_1D_1$. Hence in case II the closed series contains two quadruples. Outside of I and II there are no other cases of closed series of *real* quadruples.

The result of case II may be stated in the

THEOREM: *Four concyclic points A_1, B_1, C_1, D_1 on an equilateral hyperbola form four triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, whose orthocenters A_2, B_2, C_2, D_2 are also concyclic, and in central symmetry with $A_1B_1C_1D_1$, with respect to the center of the hyperbola. The orthocenters of $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$, $A_2B_2C_2$ coincide in the same order with A_1, B_1, C_1, D_1.*

The fact that A_2, B_2, C_2, D_2 are concyclic is independent of the fact that the points $A_1B_1C_1D_1$ are on an equilateral hyperbola. Necessary and sufficient condition is that A_1, B_1, C_1, D_1 are concyclic. It is always possible to pass an equilateral hyperbola through the points of a proper quadrangle. In fact, the pencil of conics through such a quadrangle intersects the line at infinity in an involution. Any finite point joined to this involution determines an involutory pencil, which contains at least one rectangular pair whose directions are parallel to the asymptotes of the corresponding hyperbola of the pencil. This hyperbola is therefore equilateral.

The foregoing theorem concerning a concyclic quadrangle is not new. It was first stated without proof by Steiner in a foot-note.[1] A purely geometrical proof was first given by Heinen.[2] Later on an analytic proof was published by Greiner.[3]

The question whether A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} may coincide in the same order, say, with $B_1D_1A_1C_1$ must be answered in the negative, if only real solutions are considered. According to (5) and the three similar equations such a condition would give for α_1, β_1, γ_1, δ_1 fractional values of $i\pi$, and, therefore no proper quadrangle on the hyperbola. Coincidences of less than four points may occur, but as they seem not of sufficient geometric interest, they will not be considered in this note.

REMARKS ON A PREVIOUS ARTICLE.

By NATHAN ALTSHILLER, University of Oklahoma.

In connection with my article "On the I-centers of a Triangle"[1] Prof. J. W. Clawson kindly calls my attention to the fact that the propositions (11) and (13) are known. They were proved in 1906 by the well-known Belgian mathematician, Prof. J. Neuberg, of the University of Liege.[5] Prof. Neuberg states in his article that these propositions were published before without proof.[5]

I arrived at these results in the early summer of 1917 while giving a course in "College Geometry" at the University of Oklahoma, Summer Session. The library facilities at my command were inadequate for a satisfactory biblio-

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[1] *Annales de Mathématiques Pures Appliquées*, Vol. 19, p. 43 (1828).

[2] *Journal für die reine und angewandte Mathematik*, Vol. 3, p. 291 (theorem 9) (1828).

[3] *Archiv der Mathematik und Physik*, Vol. 60, p. 184 (1877).

[4] This MONTHLY, June, 1918, pp. 241–246.

[5] *Mathesis*, 1906, pp. 14–17.

[6] These theorems were also discovered independently by J. V. Morley and published as problems in Volume 24 of this MONTHLY, pages 124 and 430.